MULTIPLIERS ON SOME TOPOLOGICAL LINEAR SPACES

在某些拓樸線性空間上之乘子

連 明 嬌

Prepared by Lien Ming-Chiao

Directed by Prof. Hang-Ching Lai

Institute of Mathematics, National Tsing Hua University

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1. Introduction and preliminaries

This paper contains three sections. Section 2 is the main part of this paper, in which we prove a theorem related to isomorphism problem; that is: Let G_1 and G_2 be locally compact groups with left Haar measures, T be an one to one, bipositive linear transformation from $L^p(G_1)$ onto $L^p(G_2)$, $1<p<\infty$, and $T(f*g) = Tf*Tg$ whenever $f*g \in L^p(G_1)$, $T^{-1}(f*g) = T^{-1}f*T^{-1}g$ whenever $f*g \in L^p(G_2)$. Then G_1 and G_2 are topologically isomorphic. In this section, we set out some notations and definitions which remain standard throughout this paper.

1.2 L^p Spaces $(1 \leq p \leq \infty)$

Let G be a locally compact group with left Haar measure dx. If G is compact, dx is assumed to be normalized so that $\int_G dx = 1$. If

G is locally compact Abelian, we denote $\hat{G}$ the dual group of G.

Let $L^p(G)$, $1 \leq p < \infty$, be the Banach space of all p-th power absolute integrable functions with respect to dx, the norm of $L^p(G)$ is given by

$$\|f\|_{L^p} = \|f\|_p = [\int_G |f(x)|^p dx]^{\frac{1}{p}},$$

and $L^\infty(G)$ denotes the Banach space of all essential bounded functions and normed by

$$\|f\|_{L^\infty} = \|f\|_\infty = \text{loc. ess. sup. } |f(x)|.$$

Denote by $C_o = C_o(G)$ the space of all continuous functions which vanich at infinity and by $C_c = C_c(G)$ the space of all continuous functions with compact supports. The topology of ⁻ is the topology of uniform convergence defined by restricting t $_o$ with the norm $\|\cdot\|_\infty$. The topology of C_c is the topology obtained by regarding C_c as the internal inductive limit of its subspace

$$C_{c,k} = \left\{ f \in C_c \;;\; \text{supp } f \subset K \right\},$$

where K ranges over all the compact subsets of G and each of C_c, k being regarded as a Banach space with the supremum norm.

We define the left and right translations by

$$\tau_a f(x) = f(a^{-1}x),$$

$$\rho_a f(x) = f(xa^{-1}).$$

1.3 Convolutions

Let $M = M(G)$ denote the space of all complex regular Borel measures on G with the weak topology $\sigma(M, C_c)$, and $M_{bd} = M_{bd}(G)$ be the subspace of M formed of those measures such that

$$\|\mu\| = |\mu|(G) < \infty .$$

For the space M_{bd}, together with this norm, is the dual of C_o. M_L^p [resp. M_R^p] denotes the measures $\mu \in M$ such that $\|\mu * f\|_p \leq$ const. $\|f\|_p$ [resp. $\|f * \mu\|_p \leq$ const. $\|f\|_p$], for all $f \in C_c$. If λ, $\mu \in M(G)$ and for any Borel subset E of G, we define

$$\lambda * \mu (E) \rightleftharpoons \int_G \mu(s^{-1}E)\, d\lambda(s)$$

Convolution of function f and measure μ by

$$f * \mu (x) \rightleftharpoons \int_G \Delta(y) f(xy^{-1})\, d\mu(y)$$

and

$$\mu * f(x) \rightleftharpoons \int_G f(y^{-1}x)\, d\mu(y) ,$$

where $\triangle$ (y) denote the modular function.

If f, g are functions, we define

$$f * g(x) = \int_G g(y^{-1} x)\ f(y)\ dy.$$

clearly,

$$\tau_a (f*g) = \tau_a f * g,$$

$$\rho_a (f*g) = f * \rho_a g.$$

1.4 Positive mappings

A mapping T from a function space into a function space is called positive, if $TF \geq 0$ almost everywhere whenever $f \geq 0$ almost everywhere. If T is an one - one onto mapping, T and T^{-1} are positive, we call T bipositive.

1.5 Multipliers

Let G be a locally compact group, X, Y be topological linear spaces of functions defined on G, then a continuous linear transformation T from X to Y is called a left (right) multiplier for the pair (X, Y) whenever $T\tau_s = \tau_s T$ ($T\rho_s = \rho_s T$) for each $s \in G$. If T is both a left and a right multiplier, then we call simply T a multiplier.

We denote the set of all left (right) multipliers for ($L^p(G)$, $L^p(G)$) by $M_\ell(L^p)$ [resp. $M_R(L^p)$] and the set of all multipliers for ($L^p(G)$, $L^p(G)$) by $M(L^p)$, $1 \leq p \leq \infty$.

Denote by $M_\ell(L^p, L^q)$ [resp. $M_R(L^p, L^q)$] the set of all left (right) multipliers for (L^p, L^q), $M(L^p, L^q)$ the set of all multipliers for (L^p, L^q), $1 \leq p$, $q \leq \infty$.

2. Isomorphism theorems relate to multipliers

2.1 Introduction

Let G_1 and G_2 be locally compact (Hausdorff) groups, and $E(G_i)$ denote the function space over the group G_i.

The isomorphism problem consists of that at what conditions on the mapping of $E(G_1)$ onto $E(G_2)$ can deduce to the isomorphic topological groups G_1 and G_2. First kawada [5] proved that if there exists a bipositive isomorphism of $L^1(G_1)$ onto $L^1(G_2)$, then G_1 and G_2 are topologically isomorphic.

Wendel [10] [11] proved the isomorphic groups from the hypothesis that if there is a norm nonincreasing isomorphism of $L^1(G_1)$ onto $L^1(G_2)$. Later Edwards [2] consider the situation where the groups $G_i(i=1, 2)$ are compact and there exists a bipositive isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leq p < \infty)$ and proved under these conditions, the groups are topologically isomorphic. In Edwards[2], he asked whether the compact groups G_1 and G_2 are necessarily isomorphic if the bipositive is replace by isometry. The affirmative answer to this question was given by Strichartz [9] . Further, Parrott [7] proved the question for general locally compact groups G_1 and G_2 under the isometric transformation of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leq p < \infty, p \neq 2)$ and some additional conditions which are necessary for the Lebesgue space L^p (Indeed, $L^p(G)$ needs not be an algebra if G is not compact). We ask that whether the Parrott's result holds if the isometry is replaced by bipositive. That is the same question that in Edwards[2], whether the locally compact groups G_1 and G_2 are necessarily isomorphic if we assume that there is an injective bipositive linear mapping from the Banach space $L^p(G_1)$ onto the Banach space $L^p(G_2)$. For this purpose we give the affirmative answer in this paper.

Some another isomorphic problems were given by Johnson [4] , Gaudry [3] and Strichartz [8] .

Johnson [4] showed on the bounded regular measure algebra under the isometric isomorphism. Gaudry [3] proved on the multiplier algebra $M(L^p)$ $1 \leq p < \infty$ under the condition of isometric isomorphism and bipositive isomorphism.

2.2 The main theorem

For convenient, we state the following

Lemma A: If T is a positive linear transformation of $L^p(G_1)$ into $L^p(G_2)$ $(1 \leq p \leq \infty)$, then T is continuous.

Proof: We give the alternative proof of Brainerd and Edwards [1] as following: If T were not bounded, these would exist $\{f_n\}$ in $L^p(G_1)$ such that

$$\|f_n\|_{L^p(G_1)} \leq 1 \qquad \text{and} \quad \|Tf_n\|_{L^p(G_2)} \geq n^3$$

Since T is positive, $|Tf_n| < T|f_n|$, and so we may assume that $f_n \geq 0$. The series $f = \sum_{n=1}^{\infty} n^{-2}$ Tf converges in $L^p(G_1)$ and $f_n < n^2 f$, we have $0 \leq Tf_n \leq n^2$ Tf and

$$n^2 \|Tf\|_{L^p(G_2)} \geq \|Tf_n\|_{L^p(G_2)} \geq n^3$$

implies $\|Tf\|_{L^p(G_2)} \geq$ n which is a contradiction for n may be large enough.

We shall need the result of Brainerd and Edwards [1; Theorem 3.5] later, thus we state as following.

Theorem B: If T is a positive linear map of L^p into $L^p (1 \leq p \leq \infty)$ which commute with ρ_a [resp. τ_a], then there exists a positive $\mu \in M_L^p$ [resp. M_R^p] such that

$$Tf = \mu * f \quad [\text{ resp. } f * \mu]$$

for $f \in C_c$; if $p < \infty$, the above identity holds for $f \in L^p(G)$. And conversely.

Now we are going to give our main theorem as following.

Theorem: Let G_1 and G_2 be locally compact groups with left Haar measures, Let T be an one to one, bipositive linear transformation from $L^p(G_1)$ onto $L^p(G_2)$, where $1 < p < \infty$, and satisfying $T(f*g) = Tf*Tg$ whenever $f*g \in L^p(G_1)$ and $T^{-1}(f*g) = T^{-1}f*T^{-1}g$ whenever $f*g \in L^p(G_2)$. Then G_1 and G_2 are topologically isomorphic.

Proof of the theorem:

As T is an one to one linear mapping of $L^p(G_1)$ onto $L^p(G_2)$, it is immediately that $T\rho_a T^{-1}$ is a linear operator on $L^p(G_2)$ for every $a \in G_1$.

If $f, g \in L^p(G_2)$ and $f*g \in L^p(G_2)$, we have $(T\rho_a T^{-1})(f*g) = T\rho_a (T^{-1}f*T^{-1}g) = T(T^{-1}f*\rho_a T^{-1}g) = f*T\rho_a T^{-1}g$.

And for $b \in G_2$,

$$\tau_b (T\rho_a T^{-1})(f*g) = \tau_b f*T\rho_a T^{-1}g,$$

and

$$(T\rho_a T^{-1})\tau_b (f*g) = (T\rho_a T^{-1})(\tau_b f*g) = \tau_b f*(T\rho_a T^{-1})g.$$

Therefore,

$$\tau_b(T\rho_a T^{-1})(f*g) = (T\rho_a T^{-1})\tau_b(f*g)$$

whenever $f*g \in L^p(G_2)$, where $a \in G_1$, $b \in G_2$.

Since C_c*C_c is norm dense in L^p, we see that $T\rho_a T^{-1}$ commutes with left translation for each $a \in G_1$.

As T and T^{-1} are positive linear transformation, we see that T and T^{-1} are bounded by Lemma A. Hence $T\rho_a T^{-1}$ is a bounded positive linear operator on $L^p(G_2)$ which commutes with τ_b, $b \in G_2$. Then there exist positive measures μ, $v \in M_R^p(G_2)$ such that

$$(T\rho_a T^{-1})f = f*\mu \qquad \text{for } f \in L^p(G_2),$$

and

$$(T\rho_{a^{-1}} T^{-1})f = f*v \qquad \text{for } f \in L^p(G_2).$$

(by Theorem B)

Since

$$(T\rho_{a^{-1}} T^{-1})(T\rho_a T^{-1})f = f*\mu*v \quad \text{for } f \in L^p(G_2),$$

and

$$(T\rho_{a^{-1}} T^{-1})(T\rho_a T^{-1})f = f = f*\delta_0 \quad \text{for } f \in L^p(G_2),$$

We have

$$\mu * v = \delta_0 .$$

Next, we prove μ, v are Dirac measures.

Suppose that b_1 and b_2 are two distinct points of the support of μ, c is a point of the support of v (b_1, b_2, $c \in G_2$). Since G_2 is Hausdorff, we can choose a neighborhood U of the identity $e_2 \in G_2$ such that $b_1 U c U \cap b_2 U c U = \phi$. choose a function $\psi \in C_c(G_2)$ with $0 \leq \psi \leq 1$ such that $\psi(e_2) = 1$ and support $\subset U$ (by Urysohn lemma) Define

$$\mu_1 = (\tau_{b_1}\psi)\mu + (\tau_{b_2}\psi)\mu \quad ,$$
$$v_1 = (\tau_c\,\psi)v \quad ,$$

then μ_1, v_1 are positive, nonzero measures, and it is obvious that $v_1 \leq v$, $\mu_1 \leq \mu$, and

$$\mu_1 * v_1 \leq \mu * v = \delta_0$$

But $\mu_1 * v_1$ is a positive measure with at least two distinct points b_1c, b_2c in its support. We can show this by the following:

$$\mu_1 * v_1(b_1c_1) = \int_{G_2} v_1(y^{-1}b_1c)d\mu_1(y)$$
$$\geq v_1(b_1^{-1}b_1c)\mu_1(b_1)$$
$$\geq \mu(b_1)v(c)$$
$$> 0$$

Similarly, $\mu_1 * v_1(b_2c) \geq \mu(b_2)\,v(c) > 0$.

Since δ_0 has only one point support, it dednce a contradiction.

Therefore μ, v are Dirac measures.

Let $\delta_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$,

then $f * \delta_a = \rho_a f$.

Since μ depends on a, we denote the support of μ by $\wedge(a)$ and the mass of μ by $\lambda(a)$, then $\wedge(a) \in G_2$, and

$$f*\mu = f*\lambda(a)\,\delta_{\wedge(a)} = \lambda(a)\rho_{\wedge(a)} f.$$

Hence we obtain

$$(T\rho_a T^{-1})f = \lambda(a)\,\rho_{\wedge(a)}f, \text{ for } f \in L^p(G_2).$$

Since $T\rho_a T^{-1}$ is a positive linear operator, $\lambda(a) \geq 0$ and $\wedge$ is a mapping from G_1 to G_2.

It remains to prove that $\wedge$ is an algebra isomorphism and a bicontinuous mapping from G_1 onto G_2.

It is obvious that $\wedge$ and λ are homomorphisms. Indeed,

$$\lambda_{(ab)}\rho_{\wedge(ab)}f = T\rho_{ab}T^{-1}f = (T\rho_a T^{-1})(T\rho_b T^{-1})f$$
$$= \lambda(a)\rho_{\wedge(a)}\lambda(b)\rho_{\wedge(b)}f = \lambda(a)\lambda(b)\rho_{\wedge(a)}\rho_{\wedge(b)}f$$

for every $a, b \in G_1$ and any $f \in L^p(G_2)$

If we take the norm $\|\cdot\|_p$ on both sides, we obtain

$$\lambda(ab) = \lambda(a)\lambda(b) \quad ,$$

and so

$$\rho_{\wedge(ab)} = \rho_{\wedge(a)}\ \rho_{\wedge(b)}$$

We want to show that $\cap$ is one to one and bicontinuous.

Let e_1 and e_2 be the identity of G_1 and G_2 respectively, and I_1, I_2 be the identity operators of $L^p(G_1)$ and $L^p(G_2)$ respectively.

Suppose that $\wedge(a) = e_2$, $a \in G_1$, then

$$T\rho_a T^{-1} = \lambda(a)\rho_{\wedge(a)} = \lambda(a) I_2$$

and $\rho_a = \lambda(a) I_1$, $\lambda(a) = 1$, $a = e_1$, $\lambda(e_1) = 1$.

This shows that $\wedge$ is an injective mapping. Actually $\lambda(a) = 1$ for all $a \in G_1$. In fact, if $\lambda(a) > 1$ for some $a \in G_1$, then by the reason of homomorphism λ, we can find a sequence $\{a_n\}$ in G_1

such that $\lambda(a_n) > n$, and since

$$\|T\|\,\|\rho_{a_n}\|\,\|T^{-1}\| \geq \|T\rho_{a_n}T^{-1}\| \geq \|n\,\rho_{\wedge(a_n)}\| = n,$$

We get

$$\|T\|\ \ \|T^{-1}\| \geq n.$$

This is a contradiction for sufficiently large n, since T and T^{-1} are bounded linear transformation. Therefore, $\lambda(a) \leq 1$ for any $a \in G_1$. On the other hand, if $\lambda(a) < 1$, then $\lambda(a^{-1}) > 1$, for $a \in G_1$. This shows $\lambda(a) = 1$ for all $a \in G_1$.

Now we show that $\wedge$ is bicontinuous. For convenient, we give an alternative proof for the continuity as Wendel [1] .

We observe that $\wedge$ is the product of the following mappings:

M_1: $a \to \rho_a$, $a \in G_1$.

M_2: $\rho_a \to T\rho_a T^{-1} = \rho_{\wedge(a)}$, $a \in G_1$,

$= \rho_{a'}$ (set $\wedge(a) = a'$, then $a' \in G_2$)

M_3: $\rho_{a'} \to a'$, $a' \in G_2$.

Evidently M_1 is continuous in the strong operator topology in $L^p(G_1)$ $1 < p < \infty$. We now prove that M_2 is continuous. Since T is bounded, If $\rho_a \to \rho_b$ in the strong operator topology, then

$$\|T\rho_a T^{-1}f - T\rho_b T^{-1}f\|_p \leq \|T\|\,\|\rho_a T^{-1}f - \rho_b T^{-1}f\|_p \to 0,$$

for $f \in L^p(G_2)$.

Hence $T\rho_a T^{-1} \longrightarrow T\rho_b T^{-1}$ in the strong operator topology, and M_2 is continuous.

Finally, we prove that M_3 is continuous. It is clear that M_3 is a homomorphism of groups of operators $\{\rho_{a'}\}$ onto G_2. Let V' be an arbitrary neighborhood of $e_2 \in G_2$, we shall construct a strong neighborhood of I_2 whose image under M_3 is contained in V'. If we can do this, then M_3 is continuous, since M_3 is a homomorphism

Let W' be a neighborhood of e_2 having finite measure δ and satisfying $W'W'^{-1} \subseteq V'$. Let $X' \in L^p(G_2)$ be the characteristic function of W', we shall show that if $\|\rho a'X' - X'\|\rho < 2^{\frac{1}{p}}\delta$, then $a' \in V'$. In fact, if $a' \notin V'$, then $W' \cap W'a' = \phi$ and $W' \cap W'a'^{-1} = \phi$, since for otherwise $W'a' \cap w' \neq \phi$ implies that there exists x such that $x \in W'a'$ and $x \in W'$, $xa'^{-1} \in W'$, and so

$$(xa'^{-1})^{-1}x \in W'^{-1}W' \subset V' \Rightarrow a' \in V',$$

this is a contradiction, similarly, $W'a'^{-1} \cap W' = \phi$.

In this case

$$\|\rho_a X' - X'\|_p = \left\{\int_{G_2} |X'(xa'^{-1}) - X'(x)|^p dx\right\}^{\frac{1}{p}} = 2^{\frac{1}{p}}\delta,$$

deduce a contradiction.

If $\|\rho_{a'}f - f\|_p < 2^{\frac{1}{p}}\delta$, $f \in L^p(G_2)$, then $\|\rho a'X' - X'\|_p < 2^{\frac{1}{p}}\delta$, where X' is constructed as above. So if $\|\rho_{a'}f - f\|_p < 2^{\frac{1}{p}}\delta$ for every $f \in L^p(G_2)$, we get $a' \in V'$ from the above discussion. This shows that for every neighborhood V' of e_2 in G_2, there is a neighborhood V of I_2 in the strong operator topology such that the image of V under the mapping M_3 is contained in V'. Hence M_3 is continuous. Therefore $\wedge = M_3 M_2 M_1$ is continuous.

At last, we prove $\wedge$ is an onto mapping. If $a' \in G_2$, evidently, $T^{-1} \rho_{a'} T$ is a positive linear operator on $L^p(G_1)$ which commute with left translations by the same argument as done above. So there exist $a \in G_1$ such that

$$T^{-1} \rho_{a'} T = \rho_a,$$

and

$$T \rho_a T^{-1} = \rho_{a'} = \rho_{\wedge(a)}.$$

That is,

$$a' = \wedge (a).$$

This shows that $\wedge$ is an onto mapping. Hence the inverse $\wedge^{-1}$ of $\wedge$ from G_2 onto G_1 exists and continuous.

From the above argument, we prove that $\wedge$ is an algebra isomorphism and a bicontinuous mapping from G_1 onto G_2. Therefore, G_1 and G_2 are Topologically isomorphic. Q. E. D.

3. Additional remarks

Remark 1. Whether the isomorphism problem holds for the space $L^{\infty}(G)$ for general locally compact froup G is still open. Note that if G is compact, the problem was given by Strichartz [9] .

Remark 2. The problem for isomorphism that are neither isometric nor bipositive would appear to remain largely open.

Remark 3. Assume that G_1 and G_2 are locally compact Abelian. $M(L^p(G_i))$ and $M(L^q(G_i))$, $i = 1, 2$, have isometric and bipositive isomorphism between them, where $\frac{1}{p} + \frac{1}{q} = 1$. Use this property, we can extend the result of Gaudry [3] such that if there is an isometric or bipositive isomorphism from $M(L^p(G_1))$ onto $M(L^q(G_2))$, $\frac{1}{p} + \frac{1}{q} = 1$, (p. q $\neq$ 2 for isometric case), then G_1 and G_2 are topologically isomorphic.

Remark 4. Assume that G_1 and G_2 are locally compact Abelian groups. $M(C_o(G_i))$ and $M(G_i)$, $i = 1. 2$, have isometric and bipositive isomorphism between them. If there exists an isometric or bipositive isomorphism from $M(C_o(G_1))$ onto $M(C_o(G_2))$, then G_1 and G_2 are topologically isomorphic.

Remark 5. Assume that G_1 and G_2 are locally compact groups. $M(L^1(G_i), L^p(G_i))$, $M(L^q(G_i), L^{\infty}(G_i))$ and $L^p(G_i)$, $i = 1. 2$, have isometric and bipositive isomorphism one another. We can extend the result of Parrott [7] and the theorem we just proved such that if T is an isometric [resp. bipositive] isomorphism from $M(L^1(G_1), L^p(G_1))$ onto $M(L^1(G_2), L^p(G_2))$ or from $M(L^q(G_1), L^{\infty}(G_1))$ onto $M(L^q(G_2), L^{\infty}(G_2))$, $p.q \neq 2$, $1 \leq p.\ q < \infty$ [resp. $1 \leq p < \infty$] , then G_1 and G_2 are topologically isometric.

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