THE SARFACE SYSTEM OF DILUTION SOLUTION OF HE^3 IN HE^4 TREATED AS A SYSTEM OF TWO DIMEUSIONAL IDEAL FERMI GAS

用二度空間佛米氣體之模型來處理 He^3 與 He^4 稀溶液的表面系統

郭 鴻 銘 (HONG - MING GUO)

Dilute solutions of He^3 in He^4 have been the subject of a good deal of theoretical and experimental research in the last several years and are reasonably well understood. They were first treated theoretically by Landau and Pomeranchuk.(1),(2) They considered the change in the liquid He^4 due to the presence of He^3 impurity atoms in terms of elementary excitations called He^3 quasiparticles which have the energy spectrum

$$E=-E_3+\frac{p^2}{2m^*} \tag{1}$$

Where $-E_3$ is the binding energy of a single He^3 atom in the ground state in He^4, m^* is the effective mass of the quasiparticle and p is its momentum. At temperatures below 0.5-0.6k when phonon and roton excitations are negligible, Edwards, et al(3) and Anderson, et al(4) showed in their heat capacity measurement experiment that these dilute system can be treated as a system of ideal Fermi gas of He^3 quasiparticles if one allowed for the effective mass varies with concentration*. The number density of He^3 quasiparticles equals to that of He^3 impurity atoms. A natural extension of ones thought along this line is that the surface of dilute He^3 - He^4 solutions might behave in a similar way to a two dimensional ideal Fermi gas of surface He^3 quasiparticles.

Andreev[5] treated the surface system of dilute He^3 - He^4 mixtures as a system of two dimensional ideal gas of surface He^3 quasiparticles with energy spectrum

$$E=-E_s-\varepsilon_0+\frac{p^2}{2M} \qquad (2)$$

where $-\varepsilon_0$ is the difference of the ground state energy of the surface state and the He^3 quasiparticle in the bulk. M can be considered as the effective mass of the surface He^3 - quasiparticle. Starting from equation (2), he calculated the surface tension σ of the system in the Boltzmann region to be

$$\sigma=\sigma_4-x\frac{\hbar\rho}{m_4}\frac{M}{m^*}\left(\frac{2\pi k_B T}{m^*}\right)^{\frac{1}{2}} e^{\frac{\varepsilon_0}{k_B T}} \qquad (3)$$

Where σ_4 is the surface tension of the pure solvent He^4, x is the He^3 concentration, m_4 is the mass of He^4 atom, ρ is the densily of liquid He^4, k_B is the Boltzmann constant, T is the temperature and $\hbar$ is Planck constant divided by 2π. He compared his results with Eselson's[6] experimental results, and later Zinoveva[7] et al did an experiment checked with Andreev's predication. The results indicated that two dimensional ideal gas is a good working model for the surface system. This work is to extend Andreev's calculation to the degenerate region. By comparing the result with experimental data in the lower temperature region, one should be able to determine whether the two dimensional Fermi gas is a good model or not.

The average occupation number $\langle n_{\vec{p}}\rangle$ of a Fermi gas is

$$\langle n_{\vec{p}}\rangle=\frac{1}{\eta^{-1}\exp\beta\epsilon_{\vec{p}}+1} \qquad (4)$$

Where $\epsilon_{\vec{p}}$ is the energy in the one particle state with momentum $\vec{p}$, $\eta=e^{\beta\mu'}=e^{\beta(\mu_3+\varepsilon_0)}$, $\beta=\frac{1}{k_B T}$, μ_3 is the He^3 chemical potential in the bulk of He^3 - He^4 mixtures. At equilibrium

μ_3 is equal to the He^3 surface chemical potential. For the surface system, the excess number of He^3 particles per unit area of the surface n_s, the surface adsorption, is then

$$n_s = \frac{2}{h^2}\int_0^\infty \frac{\eta \exp(-\beta \frac{p^2}{2M})}{1+\eta \exp(-\beta \frac{p^2}{2M})} \cdot 2\pi p dp$$

let $x=\beta \frac{p^2}{2M}$,

then $$n_s = \frac{4\pi M}{\beta h^2}\int_0^\infty \frac{\eta e^{-x}}{1+\eta e^{-x}} dx = \frac{-M}{\pi\beta\hbar^2}\ln(1+\eta e^{-x})\Big|_0^\infty$$

$$\therefore n_s = \frac{Mk_BT}{\pi\hbar^2}\ln\left[1+\exp\left(\frac{\mu_3+\epsilon_0}{k_BT}\right)\right] \tag{5}$$

From the well-known thermodynamic formula[8]

$$n_s = -\left(\frac{\partial\sigma}{\partial\mu_3}\right)_T$$

equation (5) can be written in the following form

$$\frac{\pi\hbar^2}{Mk_BT}d\sigma = -\ell n[1+e^{\beta\mu'}]d\mu' \tag{6}$$

Integrate equation (6), notice that as the He^3 concentration x approaches to zero the He^3 chemical potential μ_3 and hence μ' approaches to $-\infty$, we obtain

$$= \sigma_4 - \frac{M(k_BT)^2}{\pi\hbar^2}\int_{-\infty}^{\xi}\ell n(1+e^{\xi})d\xi \tag{7}$$

Where $\xi = \beta\mu'$ Let $\eta = e^{\xi}$ we have

$$\sigma = \sigma_4 - \frac{M(k_BT)^2}{\pi h^2}\int_0^{\eta}\frac{\ell n(1+\eta)}{\eta}d\eta \tag{8}$$

In the Boltzmann region, when temperature $T \gg T_{Fs}$, where $T_{Fs} = \frac{\pi\hbar^2}{Mk_B}n_s$ is the Fermi temperature of the two dimensional Fermi gas, it can easily be shown, from equation (5) that $\eta \ll 1$. Under this condition equation (8) can be integrated by series expansion. Neglecting the higher order terms.

$$\sigma=\sigma_4-\frac{M(k_BT)^2}{\pi\hbar^2}e^{\beta(\mu_3+\varepsilon_0)} \tag{9}$$

The He^3 chemical potential μ_3 of the bulk liquid in the Boltzmann region is (9)

$$\mu_3=-k_BT\,\ell n\left[\frac{2m_4}{xp}\left(\frac{m^*k_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right] \tag{10}$$

combining equation (9) and equation (10), we obtain Andreev's result in equation (3).

In the degenerate region when $T<T_{Fs}$ then $\eta<1$, equation (8) can be integrated as follows.

$$\sigma=\sigma_4-\frac{M(k_BT)^2}{\pi\hbar^2}\left[\int_0^1\frac{\ell n(1+\eta)}{\eta}d\eta+\int_0^\eta\frac{\ell n(1+\eta)}{\eta}d\eta\right]$$

$$\therefore\ \sigma=\sigma_4-\frac{M(k_BT)^2}{\pi\hbar^2}\left\{\frac{\pi^2}{12}+\left[\frac{(\ell n\eta)^2}{2}-\frac{1}{\eta}+\frac{1}{2^2\eta^2}-\frac{1}{3^2\eta^3}+\cdots\cdots(-1)^n\frac{1}{n^2\eta^n}+\cdots\cdots\right]_1^\eta\right\}$$

$$\therefore\ \sigma=\sigma_4-\frac{M(k_BT)^2}{\pi\hbar^2}\left\{\frac{\pi^2}{6}+\frac{(\ln\eta)^2}{2}-\frac{1}{\eta}+\frac{1}{2^2\eta^2}-\frac{1}{3^2\eta^3}+\cdots\cdots(-1)^n\frac{1}{n^2\eta^n}+\cdots\cdots\right\}$$

$$\therefore\ \sigma=\sigma_4-\frac{M}{2\pi\hbar^2}(\mu_3+\varepsilon_0)^2\left\{1+\frac{\pi^2}{3\xi^2}-\frac{2}{\xi^2}e^{-\xi}+\cdots(-1)^n\frac{2}{n^2\xi^2}e^{-n\xi}+\cdots\cdots\right\} \tag{11}$$

In the highly degenerate region, one can keep only the first few leading terms in the bracket. With the dilution refrigerator available commercially which can operate in the milidegree range continuously, it is not a difficult task to measure the surface tension in the degenate region. ($T\lesssim 0.2^\circ k$ for most dilute concentotion). By comparing with the experimental data one can determine the validity of this model and determine the two parameters M and ε_o in equation (2). The presence of weak interactions between the surface quasiparticles will show up on the dependence of M on the concentration x. The same result can be obtained by explaining the surface tension σ as the surface pressure(10) of the two dimensional Fermi gas and using the relation between the pressure and the grand partition function of the system.

* Experiment showed that the value of M* depends on the He^3 concentration. This indicates that a weak interaction exists between He^3 quasiparticles.(11)(12)

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